

Chapter 2

The $nsm(x)$ Diagrams

The structures in this book are composed of beams and bars (and cables). These are long and relatively slender line elements which have a longitudinal axis passing through the centers of area of the cross-sections. The coordinate along the axis is x .

On both sides of virtual sections perpendicular to the axis we find equal and opposite internal forces: normal forces n acting along the axis, shear forces s perpendicular to the axis (along the faces of the section), and equal and opposite couples, the bending moments m . Calculating these internal forces throughout the structure is the main purpose of structural analysis.

This chapter shows how to compute and present the internal forces along any line element if the external (applied) forces are all known. The internal forces are functions of x , leading to the normal force distribution $n(x)$, the shear force distribution $s(x)$ and the bending moment distribution $m(x)$ or diagrams, also called $nsm(x)$ or $\mathbf{n}(x)$ for short.

These distributions are drawn along the elements in a traditional manner, and we will show how to compute and draw these distributions or diagrams.

2.1 Line Elements

The essence of the analysis of a structure is the computation of its internal forces: the normal force $n(x)$, the shear force $s(x)$ and last, but by far not least, the bending moment $m(x)$ at any section x along the structure. We will refer to these quantities also in their vector form $\mathbf{n}(x) = \{n(x) \ s(x) \ m(x)\}^T$ and also symbolically as $nsm(x)$.

The element is represented by the element axis x which is assumed to follow a line on the center of mass of the cross-sections. The line need not be straight; it can be moderately curved or kinked or composed of rigidly connected segments. The line must be open; no closed loops are allowed at this point. To begin, we consider lines with one starting point and one end point (Fig. 2.1).

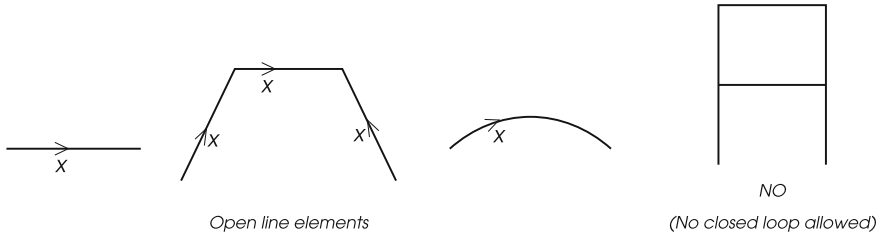


Fig. 2.1 Line elements

The ‘floating’ line elements will be subjected to concentrated or point forces and couples, and to uniform distributed forces but not to distributed couples. These loads will always be self-equilibrating.

2.2 Positive, Negative nsm

Structural analysis is based on the premise that if you could pry open a structural element by a cut perpendicular to the axis, you would discover, on both sides of the section, equal and opposite internal forces. The facet to the left is called the negative or $-$ facet and the facet to the right is the positive or $+$ facet.¹ The plethora of forces on the facets can be reduced to statically equivalent equal and opposite normal forces n and shear forces s positioned at the center of area of the cross-section and corresponding equal and opposite couples, called bending moments or simply moments m .

When the nsm are as depicted in the first row of Fig. 2.2 they are said to be positive. If they have opposite sense (second row) they are said to be negative.²

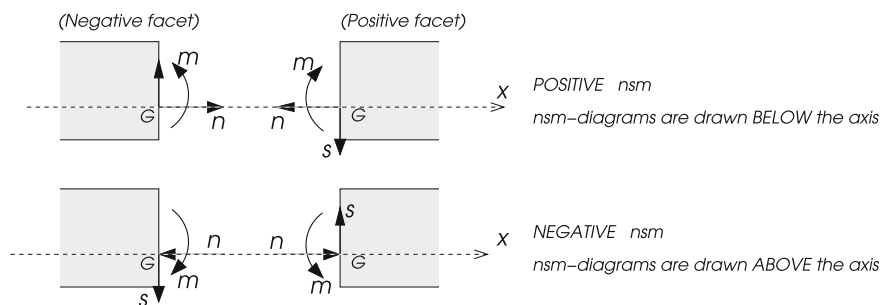
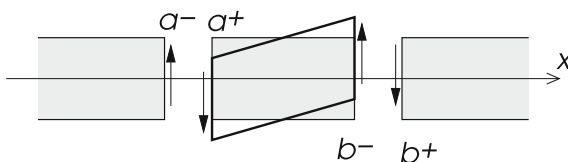
Physically, a positive normal force n causes tension at that location, a positive moment sets the lower fibers in tension and the upper fibers in compression, and positive shear would deform a segment as depicted in Fig. 2.3. Assuming that in the depicted beam segment the shear is positive, the shear forces at two neighboring sections a and b will look as shown (everything there is positive shear).

Having computed the internal forces along x , they are presented in the form of a normal (force) diagram $n(x)$, a shear (force) diagram $s(x)$ and a (bending) moment diagram $m(x)$. Computing the internal forces and displaying them are two separate albeit related activities.

The computational aspects of $nsm(x)$ are practically universal. We will present some essential computational techniques although, in due time, you will certainly

¹Here we deliberately digress from Solid Mechanics convention where ‘positive’ depends on the direction of the outside normal and the left facet is therefore the positive facet. It would be too confusing in this text.

²Do not attach too much importance to positive or negative here. Different authors, industries and countries have different signs for these forces.

**Fig. 2.2** *nsm* sign convention**Fig. 2.3** Positive shear

develop your own personal methods. As long as they produce correct results they are valid techniques.

Displaying the diagrams is another matter. In a diagram of a function $f(x)$ for instance, the coordinate x usually runs from left to right and, positive $f(x)$ are usually depicted above the x axis. This is not so for the internal forces. It varies from country to country and often from discipline to discipline. Now, whether you draw a positive moment above or below the line depends on the specific environment you are working in. Since there are two alternatives for every diagram, and three diagrams in total, there are eight ways to depict the $nsm(x)$ diagrams.

In this text we have opted for one out of these eight possibilities.

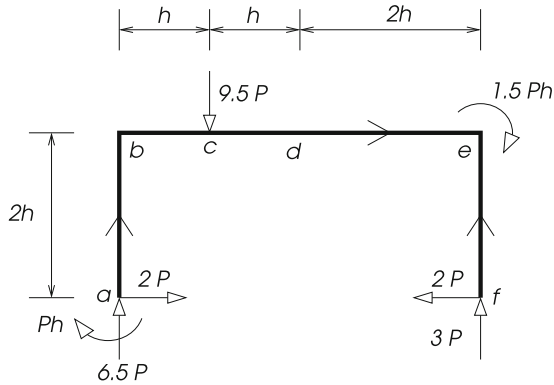
POSITIVE *nsm* (upper row in Fig. 2.2) are drawn BELOW the axis.

If you find yourself in an environment or browsing through a textbook with a different protocol for the diagrams, you will no doubt adapt to that specific presentation.

2.3 Calculating *nsm* at a Section

To compute the internal forces *nsm* in an element, we need to know all the external loads applied to that element. These must of course be in equilibrium. (If you attempt to compute the internal forces in a floating structure subjected to forces which are not auto-equilibrated you will obtain nonsensical results.)

The frame represented by line $a..f$ in Fig. 2.4, for instance, is subjected to loads in equilibrium. (When properly supported this structure is called a *portal frame*.)

Fig. 2.4 Frame-like line

You may check that the applied loads are auto-equilibrated, in other words, the loads are statically equivalent to zero (to nothing).

To compute the internal forces nsm_d at mid-span of segment be , for instance, we perform a virtual cut at d thus creating two segments ad^- and d^+f . Enforcing equilibrium on ad^- will give the internal forces nsm_{d-} . Alternatively, requiring equilibrium of d^+f will produce nsm_{d+} . (The positive sense of x along the elements is indicated by the arrows.) This is classical structural mechanics. However, we would like to advocate one specific technique to calculate the internal forces. It is based on considering the left segment (ad^-) and calculating loads at d^- which are statically equivalent to the external loads on segment ad^- .

- **Equivalent loads at d^- :** In Fig. 2.5a we consider segment ad^- of the frame including the external load applied to the segment. We compute new external loads at d^- which are statically equivalent to the applied loads (Fig. 2.5b). This is done in two steps.

In the first step we move the applied loads to d^- , including couple $-Ph$ (assuming positive couple counterclockwise).

In the second we add to couple $-Ph$ the moments with respect to d^- of the applied forces, that is $2P \times 2h - 6.5P \times 2h + 9.5P \times h = 0.5Ph$, which gives the equivalent loads shown in Fig. 2.5b.³

- **nsm_{d-} :** Clearly, to equilibrate ad^- with loads at d^- all we need to do is apply loads equal and opposite to the equivalent loads (black arrows in Fig. 2.5c). These are the nsm_{d-} .
- **nsm_{d+} :** The nsm_{d+} are equal and opposite to nsm_{d-} (black arrows in Fig. 2.5d). It turns out that the nsm_{d+} (Fig. 2.5d) are identical to the equivalent loads (Fig. 2.5b).
- **nsm_d :** This notation refers simultaneously to the internal forces on both facets of section d .

³The equivalent couple can, in principle, be anywhere on the segment and not specifically at d , but since the bending moment is at d we also place the couple of the equivalent system there.

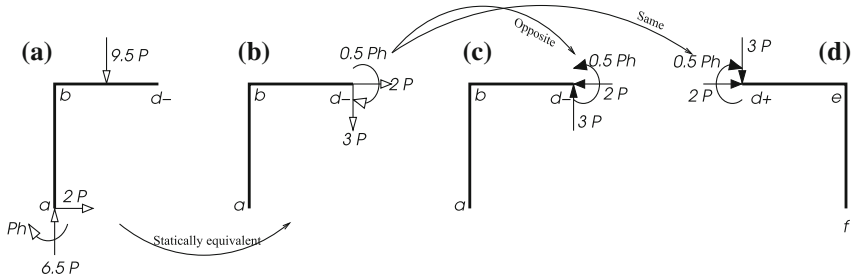


Fig. 2.5 Calculating nsm at a fictitious section. **a** Loads on segment abd^- . **b** Equivalent loads at d^- . **c** nsm at d^- . **d** nsm at d^+ (Below the axis in nsm diagrams)

In conclusion, if you want to determine nsm at some section d take the segment from the left end until d and compute the equivalent loads at d . These loads are the nsm_{d^+} , that is, the internal forces applied to the $+$ facet of d . An important fringe benefit of computing the internal forces at d^+ is that the shear force s_{d^+} points in the direction of where the shear is drawn, as we shall see in the sequel.

2.4 The nsm Diagrams

The values of the internal forces are needed not only at one section but usually everywhere along the structure. Repeating the above procedure for all the points along the structure may seem a daunting task but, as we shall see, shortcuts exist.

The internal forces are displayed in three separate diagrams: the normal, shear and moment *diagrams or distributions*. The diagrams are in fact graphs $n(x)$, $s(x)$ and $m(x)$ where the abscissa x is a line which follows the layout of the elements, and the ordinates, which are always perpendicular to the x -line, carry the respective nsm values.

2.4.1 Basic Rules

The first two *rules* for drawing the graph of the nsm functions, as for any function $f(x)$, are to define the direction of the x axis and the direction of the ordinate axis.

1. First we define the sense of the axis of the element. The running variable is denoted by x . If the element is horizontal or almost horizontal, it will run from left to right. If the element is vertical (or almost vertical), it will usually run from bottom to top, unless otherwise specified. (Remember, no axis sense, no diagrams!)

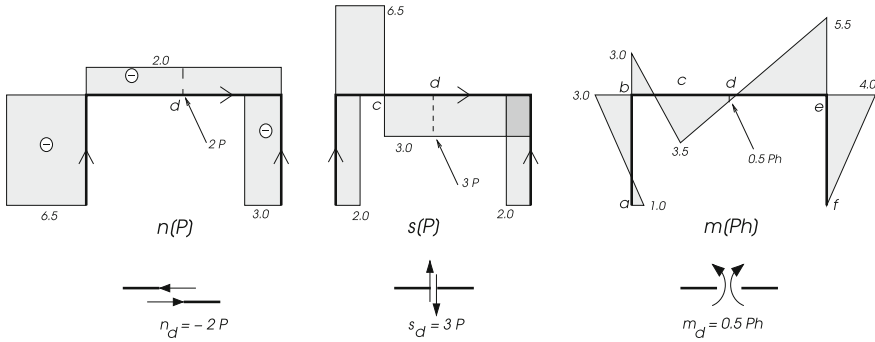


Fig. 2.6 Example of nsm diagrams

2. The positive ordinate axes of the $nsm(x)$ functions point *downwards*. This is akin to drawing a function $f(x)$ with the abscissa x running horizontally from left to right and the ordinate f pointing down. In other words, positive nsm (see Fig. 2.2) are drawn below the x axis. Negative nsm are drawn above the x axis. If the axis of the element is curved, as for an arch, the normal force is the component of the internal force (locally) tangent to the axis and the shear force is the component of the internal forces (locally) perpendicular to the axis. The ordinates of the respective nsm functions are also drawn perpendicular to the (curved) element axis.

In Fig. 2.6, we show the nsm diagrams for the frame in Fig. 2.4. We note that the diagrams indeed conform with the values nsm_d mid-way on beam be (the dashed line in the diagrams). The normal force $n_d = -2P$ being negative appears above the axis, the shear force is positive and is therefore drawn below the axis, and the bending moment is positive and is also drawn below the axis. The units of the diagrams appear in square brackets below each graph.

You will have noted that we have added a ‘-’ character in the boxes of the normal force. This is because the conventions for the sign of the normal force are poorly standardized and the difference between tension and compression normal force can be a matter of survival of the structure (in compression slender members may buckle). To avoid any misunderstandings it is good practice to add a ‘-’ for compression members and a ‘+’ for tension members. Even this may be ambiguous, so you may add any accepted notation to make sure that the difference between compression and tension is clear, such as C , T for instance.

2.4.2 Practical Guidelines for Positioning the s and m Diagrams

The rules for positioning the nsm diagrams are rather clear. Positive nsm are below the axis. But you still have to remember what positive means.

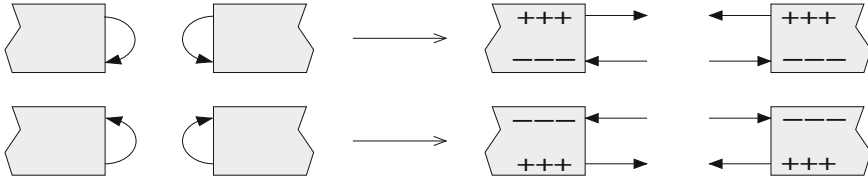


Fig. 2.7 Conventions for drawing the bending moment

So here are two simple guidelines for the *m* and *s* diagrams respectively:

- The *bending moment* is drawn on the side of the fibers in *tension*.

Consider the *bending moment* in Fig. 2.7. We remember that the curved arrows represent couples, the simplest representation of which are two parallel forces. One set of equal and opposite forces pulls on the facets and sets the material behind it in tension. The other set of forces pushes, causing compression. So in bending, one part of the cross-section is in tension and the other in compression. *The moment is drawn on the part of the beam where the tension is.* In the first row of Fig. 2.7 the upper fibers of the beam are in tension (+) and the lower part in compression (-). The moment will be drawn on top of the beam along the fibers in tension. In the second row of Fig. 2.7 the lower fibers are in tension and the diagram will appear below the axis.

In Fig. 2.6 at section *d* the fibers in tension are in the bottom part. The bending moment at *d* is therefore below the line. (See also Fig. 2.5c.)

- The *shear diagram* is drawn in the direction of the shear force on the + facet

(the facet to the right). For the *shear force*, look at the direction of the shear applied to the *far*-facet. It points to where the shear should be drawn. If the *shear force to the right goes up*, the shear is drawn *above* the line. If it points down, the shear is below the line.

In Fig. 2.5 the shear force at (*d*) on the + facet points down. The shear diagram at *d* is therefore below the line.

2.4.3 Relations Between the *s* and *m* Diagrams

The equilibrium equations of a lamella of a straight (or slightly curved) beam are

$$\frac{ds}{dx} + q = 0 \qquad \frac{dm}{dx} + s = 0 \qquad (2.1)$$

We recall that, assuming a coordinate system $xy\theta$ where x is the beam axis running from left to right, y points up and θ is the rotation, positive from x to y (counter-clockwise), the $ns m$ internal forces on the $-$ facet of a section are in the direction of $xy\theta$ and the distributed force q is positive in the y direction.

1. *The shear force is minus the tan of the slope of the moment diagram, and the distributed force q is minus the tan of the slope of the shear diagram.* Indeed, $dm/dx + s = 0$ gives $dm/dx = -s$ (and similarly $ds/dx = -q$).

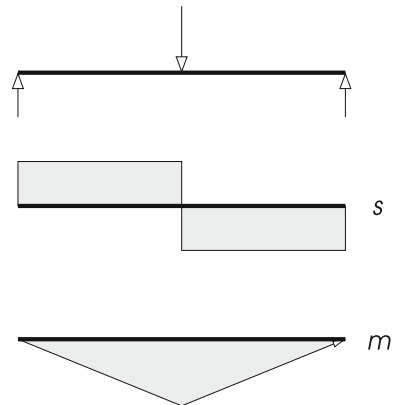
The slope of the bending moment distribution $dm/dx = -s$ thus equals minus the value of the shear force at that point. To make any use of this relation, we need to remember the positive directions of the ordinates of the s and m diagrams, and that the slope is the negative of the shear. This is not easy. It is useful to keep one typical example in mind and use it for relating the slope of the moment to the shear.

This example is the classical beam with a force in the middle of the span supported by end-forces (Fig. 2.8). The corresponding shear and moment diagrams encapsulate the entire idea. A *shear above* the line (left part) has the *moment sloping down* and a shear below the line corresponds to a moment sloping up. This is the correspondence. Remember this and you will be safe.

We have intentionally avoided using the words ‘positive’ and ‘negative’. It would only confuse things, and a picture is worth a thousand words.

2. *A segment with a constant distributed lateral force has a linear shear function and a parabolic moment function.* The shear force is linear because it has a constant slope. Taking the derivative of the second equation in (2.1) gives $d^2m/dx^2 + ds/dx = 0$ and introducing the first equation of (2.1) yields $d^2m/dx^2 = q$. Integrating this relation twice produces the algebraic equation of a parabola.
3. *The moment is locally an extremum (either maximum or minimum) where the shear force is zero.* Indeed, a local extremum of a function corresponds to a zero

Fig. 2.8 Relation between s and the slope of m



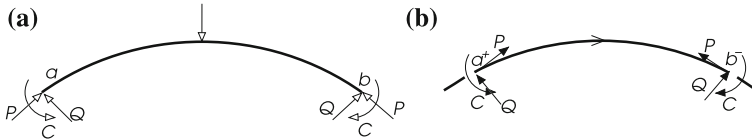


Fig. 2.9 End loads and end internal forces

slope of the function, and the slope of the moment function $dm/dx = -s$ is zero when $s = 0$.

4. A segment without distributed forces has a constant shear force and a linear moment function. This is self-evident. If the shear happens to be zero over a segment the moment is constant along that segment.
5. The difference in the values of the bending moment between two sections along a segment is equal to the area under the shear diagram between these two sections. Multiplying $dm/dx + s = 0$ by dx gives $dm + s dx = 0$. When we integrate this differential equation over the interval $[x_a, x_b]$ with $(x_b > x_a)$ we obtain $m_b - m_a + \int_{x_a}^{x_b} s dx = 0$ (dm is a total differential). Consequently

$$m_b = m_a - \int_{x_a}^{x_b} s dx \quad (2.2)$$

In other words, if you know m_a , the integral of the shear function gives you m_b .

6. The internal forces on a section very close to an extremity are identical to the applied loads at that extremity. This is obvious but not always clear to the novice, who may be tempted to draw a moment diagram with a non-zero bending moment at an extremity without an applied couple (*creatio ex nihilo*).

In the symmetric arch, symmetrically loaded, shown in Fig. 2.9a, the external loads at extremities a and b are forces P and Q and a couple C . The forces are conveniently expressed in the tangential and lateral directions. The internal loads on facet a_R^+ (a_R is a section to the right and infinitely close to a) are of course identical to the external loads at a (see Fig. 2.9b) and the internal loads on facet b_L^- (b_L is a section to the left and infinitely close to b) are identical to the external loads at b . The internal forces are thus $\mathbf{n}_a = \{-P \ -Q \ -C\}^T$ and $\mathbf{n}_b = \{-P \ Q \ -C\}^T$.

2.5 Typical Diagrams

The readers are reminded that, as a rule, straight *beams* will not have external axial forces and, therefore, also no normal internal forces. So, for beams we consider only *sm*-diagrams.

Beam with Point Force Along its Span

In Fig. 2.10 we have an example of a beam with basic auto-equilibrated forces: a point force along the span equilibrated by end forces.

The beam is composed of two unloaded segments $a_R b_L$ and $b_R c_L$, both starting immediately after a point force (subscript ‘ R ’, to the right) and ending immediately in front of a point force (subscript ‘ L ’, to the left). Note,

we NEVER make a virtual section exactly at a point-load.

We can make a virtual cut either in front or after a point load, but one cannot split a load in the same way as one cannot split a point. In Fig. 2.10 for instance, force Q is at b , and sections b_L and b_R are respectively in front and after b . Recall that the default sense of a horizontal axis is from left to right.

Since the segments are not loaded (without distributed forces), along every segment the shear is constant and the bending moment linear.

For the *shear diagram* we need *one* value along $a_R b_L$ and *one* value along $b_R c_L$. There are many ways to come by these values. We will in due course use most of them.

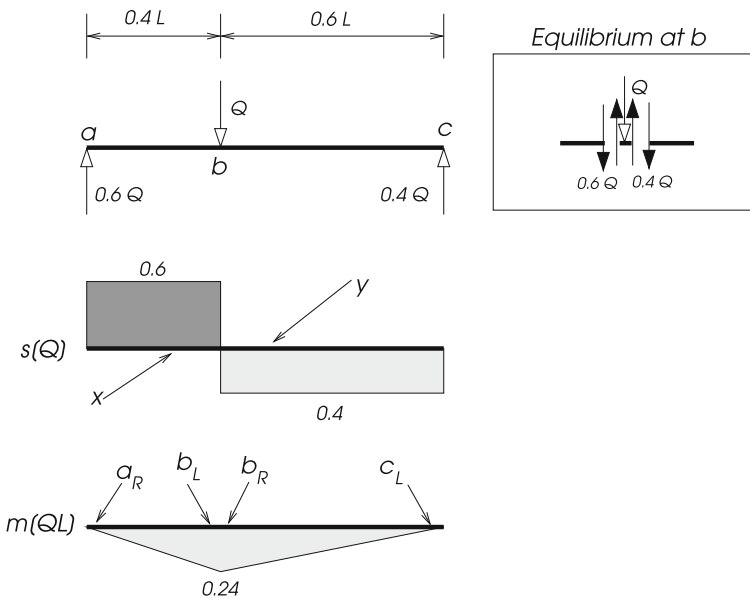


Fig. 2.10 Shear and moment diagrams for point force

- (a) For the shear at some point x on the first segment (see Fig. 2.10) we collect all the forces along ax^- (only one up force $0.6Q$). We have thus a shear of $0.6Q$ (up) at x^+ . When the shear goes *up on the right facet* it is drawn *above the axis*, hence the box on ab is above the axis (see Fig. 2.10).
(We could have used the property that the shear force on the $+$ facet near the left extremity is the applied ‘shear’, that is, the component perpendicular to the axis of the force, at that extremity.)
- (b) For the shear at y on $b_{RC}L$ we consider a section at y (Fig. 2.10), sweep over all the forces on segment ay^- , and their algebraic sum is the shear at y^+ . In our case we find a down force of $0.4Q$ from where it follows that the shear is drawn below the axis.

For the *moment diagram* we remember that the moment is linear on a_Rb_L and on $b_{RC}L$. To position a line we need its ordinates at two coordinates. For the bending moment along the first segment, we use points a_R (right after $0.6Q$ at a) and b_L (right in front of force Q at b), and for the bending moment along the second segment, we use b_R (right after force Q at b) and c_L (right in front the force at c).

You will have noticed the change in slope of the bending moment at b . It follows the sudden change in the value of the shear at that section.

Finally, in the box to the right of the figure you see the shear forces which equilibrate Q at b : forces $0.6Q$ at b_L and $0.4Q$ at b_R .

Note. We would be satisfied with these two forces but the beam could not achieve this feat without bending moments along its entire span, as evidenced by the m diagram in Fig. 2.10. These unwanted internal couples cause flexion of the beam and are sometimes the cause of beam failure.

Beam with Point Couple Along Span

The counterclockwise point couple QL along the span of the beam in Fig. 2.11 is equilibrated by the clockwise couple composed of the equal and opposite forces Q at its extremities.

Interestingly, the shear force is constant along the entire span of the beam (Fig. 2.11). It is easy to see that

a shear diagram is insensitive to point couples.

If we replace the point couple QL icon by what it represents, two equal and opposite very large forces very close to one another, such that they create a couple QL , a sweep through the couple may cause a blip in the shear diagrams (Fig. 2.12). But since the forces are infinitesimally close to one another it is inconsequential.

For the moment diagrams, we note that on both segments the moments are linear (no distributed forces) and have the same slope (same shear) but there is a discontinuity in the moment diagram at b . Clearly, close to the free ends (a_R and c_L), the bending moments are zero because there are no applied couples at the extremities. From Fig. 2.11 we conclude that $m_{b_L} = 0.4QL$, and immediately after the couple we have $m_{b_R} = -0.6QL$, hence the moment diagram.

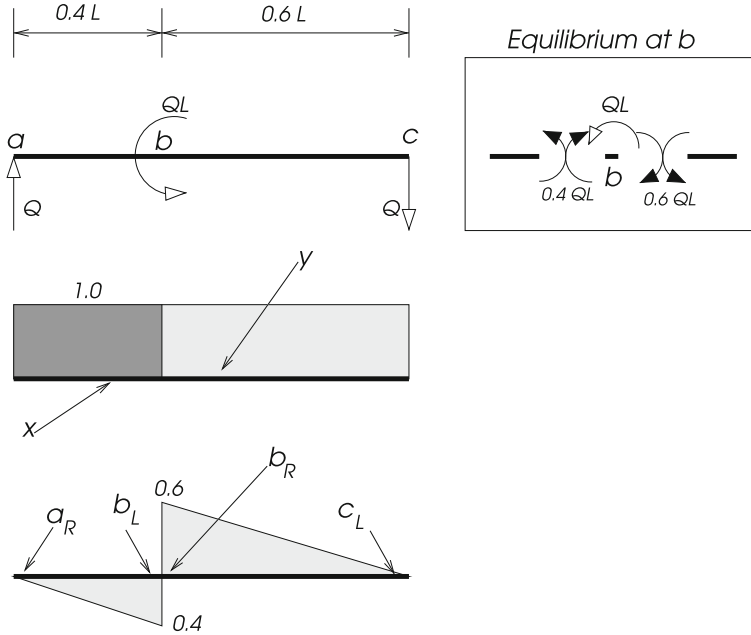
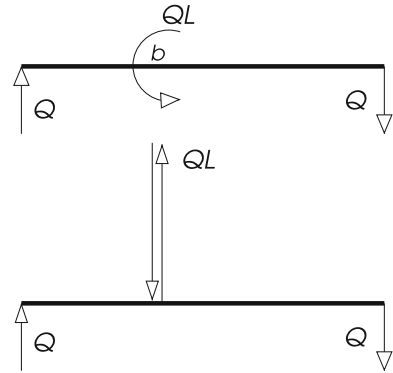


Fig. 2.11 Shear and moment diagram for point couple

Fig. 2.12 Point couple



Note, m_{b_L} could also have been obtained from the relation $m_{b_L} - m_a = -\int_{x_a}^{x_b} s \, dx = -0.4L \times -Q = 0.4QL$, that is, the area of the shear diagram over the interval (shaded area in shear diagram of Fig. 2.11). The relation is not valid for m_{b_R} because of the point couple of which the shear diagram is not aware.

Finally, the point counterclockwise couple QL at b is equilibrated by the clockwise bending moments $0.4QL$ and $0.6QL$ on both sides of b (see box in Fig. 2.11).

Beam with Uniform Distributed Forces Along its Span

The beam in Fig. 2.13 is subjected to uniformly (constant) distributed forces Q/L equilibrated by symmetric forces $0.5Q$ at the extremities. Consequently, the shear force is linear and the bending moment parabolic. For the shear force we need two values along the span. For parabolic bending moment distributions, we will require the end values, extremum values along the span (if applicable) and any other values necessary to draw a decent free-hand approximation of the distribution.

Now, at the extremities the shear forces and bending moments are equal to the applied forces and couples. This is enough to draw the shear diagram.

We notice that the shear is zero at mid-span (point b). The bending moment will have there an extremum which we must evaluate. An easy way is to use $m_b - m_a = -\int_{x_a}^{x_b} s \, dx = -0.5L \times (-0.5Q)/2 = 0.125QL$ ($m_a = 0$), that is, the area of the shaded triangle.

Note. The example deals with a symmetric beam under symmetric loading and still the shear diagram is anti-symmetric. This stems from the fact that the shear diagram is biased. It shows the shear on $+$ facets at any virtual section.

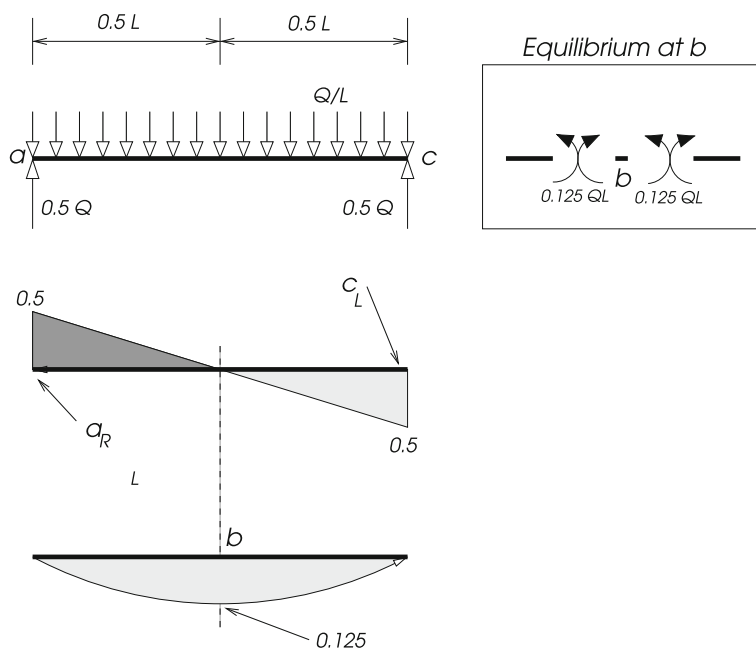


Fig. 2.13 Shear and moment diagram for uniformly distributed forces

2.6 Frames

2.6.1 Point Force on a Symmetric Frame with Inclined Columns

Frame $abcde$ in Fig. 2.14a is symmetric with respect to the vertical line passing through c and is also loaded in a symmetrical fashion by self-equilibrating forces. So each internal force nsm in segment ce is a mirror image of a corresponding nsm in segment ac . This symmetry includes the n and m diagrams but not the s diagram, as we shall see.

First, we define the positive x -axis on the elements of the frame (see Fig. 2.14b). In this case we keep the same sense of x for the entire frame.

Segments ab , bc , cd and de (taken on intentionally inbetween forces and corners) are without distributed forces and consequently the $n(x)$ and $s(x)$ distributions are constant on every segment, and $m(x)$ is linear. For the sm distributions along leg ab , we use conveniently the axial and lateral projections of $0.5Q$, respectively $0.433Q$ and $0.25Q$, shown in Fig. 2.14b. The nsm distributions are shown in Fig. 2.15.

The structure and external loading are symmetric and so are the shear forces inside the elements (see Fig. 2.16c). However, the shear diagram is not (Fig. 2.15). As you

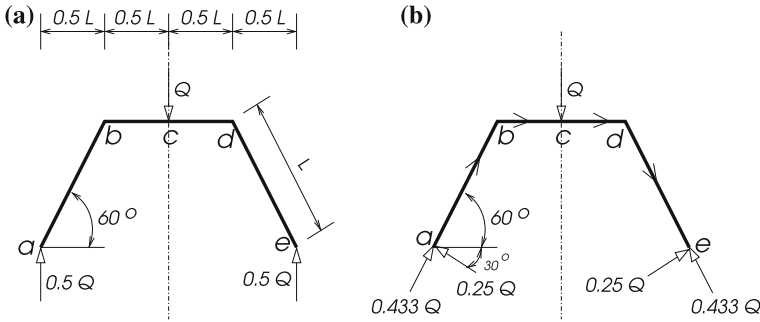


Fig. 2.14 Symmetric frame with point forces

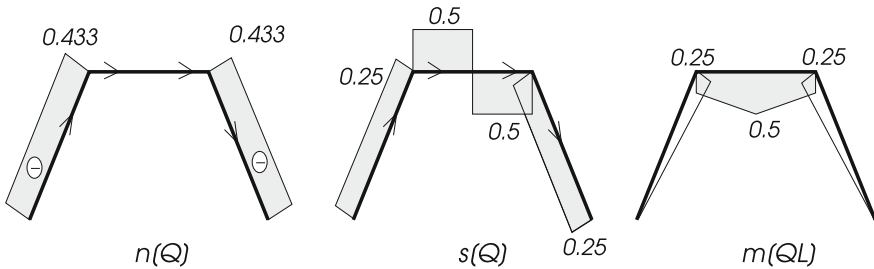


Fig. 2.15 nsm distributions in a symmetric frame

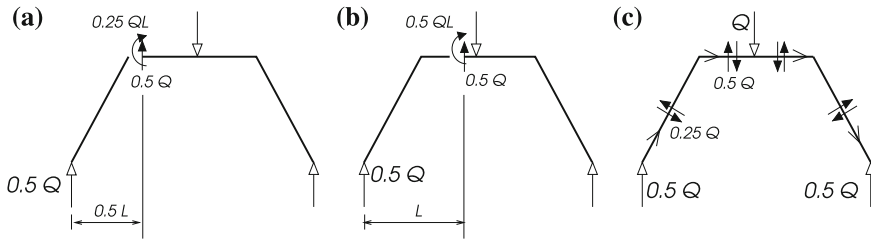


Fig. 2.16 a, b Calculating nsm ; c symmetry of s

probably realize, this is because for the shear diagram we are always looking at the $+$ facets. For a correct mirror image we need to compare the internal shear forces on the $+$ facet of the left half of the structure with those at the $-$ facet of the right half of the structure.

2.6.2 Discontinuities at Point Loads

In our view, structures are essentially subjected to zones of distributed forces with, here and there, concentrated forces and couples at the zone boundaries. Forces cause discontinuities in the n and s diagrams, and couples create discontinuities in m diagrams, as was established earlier when considering equilibrium. This was observed, for instance, in the s diagram of Fig. 2.10 and in the m diagram of Fig. 2.11.

Note, since the shear is a measure of the slope of the bending moment ($dm/dx + s = 0$), a lateral force also causes a sudden change in slope of the bending moment (see cross-section b in the bending moment diagram in Fig. 2.10).

On the other hand, a concentrated couple does not affect the shear diagram(!). In Fig. 2.11 there is a concentrated couple at b and the moment has a jump there. The shear force, however, is oblivious of the presence of the couple, and passes through that point as if there were nothing there. The slope of the bending moment is thus also not affected by the couple (the slope is $0.4QL/0.4L = Q$ on ab and the same $-(-0.6QL)/0.6L = Q$ on bc).

We would like to show one example of a discontinuity in the normal distribution. This kind of discontinuity can occur in this text only in frames as, for instance, the two-legged structure shown in Fig. 2.17a.

We will, in this case, divide the structure into *nodes* (where point loads are applied or at corners) and *elements* (segments in between nodes). This leads to the exploded view depicted in Fig. 2.17b. In the figure, the white arrows are external forces and the black arrows are internal forces. Starting from a , we impose sequentially vertical equilibrium at the nodes and at the elements. (We postpone rotational equilibrium for clarity's sake.)

Assuming positive forces upwards, the $0.75P$ force at a is balanced by $-0.75P$ on facet a_R^- , which causes $0.75P$ on facet a_R^+ of element ab . This applies an equal and

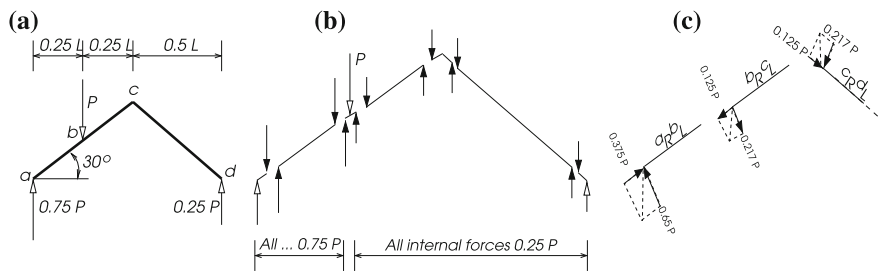


Fig. 2.17 Two-legged frame

opposite force on facet b_L^- of node b . Node b has now $0.75P$ and $-P$ (the external force) so we need to assume that facet b_R^- has $0.25P$ and so on until extremity c . (Sections b_L and b_R are infinitely close to b .) As emphasized in the drawing, all the internal forces from a to b_L are of magnitude $0.75P$ and the internal forces on the remainder of the structure have magnitude $0.25P$.

For the nsm diagrams, we recall that we are considering the three segments in between the point forces, $a_R b_L$, $b_R c_L$ and $c_R d_L$. In preparation of the n and s , diagrams we take conveniently the normal and lateral components of the internal forces at the left end of the three segments (see Fig. 2.17b).

For the m diagram we need to compute the bending moments at b and c only. (The bending moments are zero at a and d and are continuous at b and c .)

We leave it to the reader to show that rotational equilibrium of segment $a_R b_L$ requires a counterclockwise couple $0.75P \times 0.25L = 0.1875PL$ at b_L^- (see Fig. 2.18). Similarly, rotational equilibrium of segment $c_R d_L$ requires a clockwise couple $0.25P \times 0.5L = 0.125PL$ at c_R^+ . The results in Fig. 2.18 show a discontinuity both in the n and s diagrams at b . The m diagram is continuous but has a slope discontinuity at b .

The following Table 2.1 is a summary.

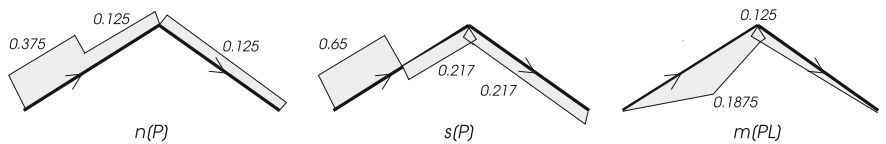


Fig. 2.18 nsm in the two-legged frame

Table 2.1 nsm discontinuities

Diagram	n	s	m
Axial force	Jump	—	—
Lateral force	—	Jump	Sudden change in slope
Couple	—	— (!)	Jump

2.7 Inverse Analysis

In broad outline, an inverse problem is when you are given the answer and you are asked for the question. Interestingly, such problems appear very often in engineering practice and research, such as when you are given the vibration profile of a structure and you want to learn something regarding the excitation.

Inverse problems are sometimes easier to come by than direct ones. For example, the standard equations of the displacement method (to be seen later on) are $\mathbf{K}\mathbf{u} = \mathbf{p}$ where $\mathbf{K}$ is a square matrix, called stiffness matrix, $\mathbf{u}$ is the nodal displacements vector and $\mathbf{p}$ is the vector of applied loads at the nodes. The direct approach is to compute the displacement, given the loads. This involves the solution of the linear system $\mathbf{K}\mathbf{u} = \mathbf{p}$.

The corresponding inverse problem would be, knowing the displacements $\bar{\mathbf{u}}$, what are the applied loads? This problem is much easier to solve. Simply multiplying $\mathbf{K}$ by $\bar{\mathbf{u}}$ yields $\mathbf{p}$

$$\mathbf{K}\bar{\mathbf{u}} \rightarrow \mathbf{p} \quad (2.3)$$

With this knowledge we realize that it can be much easier to determine the loads applied to a line element when we are given the bending moment distribution (inverse problem) than the other way around (direct approach). The rationale is clear. Ignoring the discontinuities, given $m(x)$ the first derivative produces the shear $s(x) = -dm/dx$ and the second derivative the distributed forces $q(x) = d^2m/dx^2$. In the direct approach we are dealing with integrals.

The purpose here is mainly didactic. We are, after all, learning to understand the workings of the $nsm(x)$ distributions.

Example

Consider the bending moment distribution shown in the box in Fig. 2.19. On the left half $m(x)$ is linear and on the right half $m(x)$ is a parabola starting at $-2QL$ at b with a zero value and a zero slope at c . What are the applied loads?

- Segment ab has no distributed loads but couples $5QL$ and $4QL$ at a and b respectively. This entails two equal and opposite forces $18Q$ for rotational equilibrium $5QL - 18Q \times L/2 + 4QL = 0$ (Fig. 2.19a').
- Due to the parabolic $m(x)$ segment bc has distributed forces of magnitude, say q . The applied loads are a couple $2QL$ at b and a force $-qL/2$ at mid-span, statically equivalent to q over $L/2$ (Fig. 2.19a''). This entails a force $qL/2$ at b for vertical equilibrium. (Note, the shear at c is zero because the bending moment has there a horizontal slope.) Rotational equilibrium requires $2QL - qL/2 \times L/4 = 0$ from where we find $q = 16Q/L$.

Hence the shear distribution in Fig. 2.19b and the applied loads (the purpose of this exercise) in Fig. 2.19c.

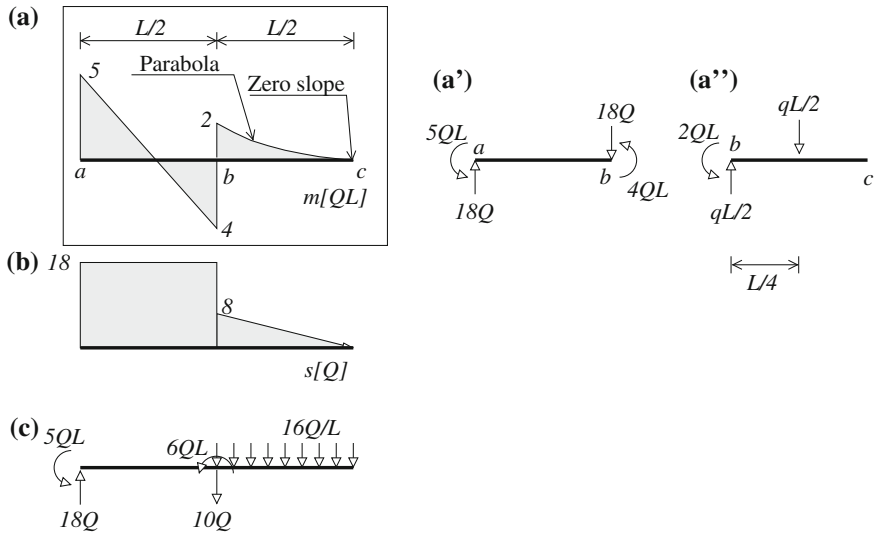


Fig. 2.19 Beam inverse analysis

2.8 Notes

We elaborate here on some notions related to the above examples, not necessarily in their order of importance.

2.8.1 You Cannot Split a Point

You will have noticed that we perform sections in front or after point loads or corners. It is not recommended to perform a section right at a corner or at a load because you cannot split a point. Simply stay away from these problematic cross-sections and be either in front or behind them. Needless to add that, however localized they may be, loads are in fact continuous forces of high magnitude distributed over a very short distance. Whatever goes on in the beam right under these forces or at corners, for that matter, is a complicated affair and belongs more to the theory of elasticity and continuum mechanics, so we had better steer clear of these regions in structural analysis. We should either be in front or behind the troublesome zones.

2.8.2 What Is in a Name?

At corners of frames without applied forces, such as b in Fig. 2.6, the ‘physical’ internal forces at b_L and b_R are the same (and opposite), only their components

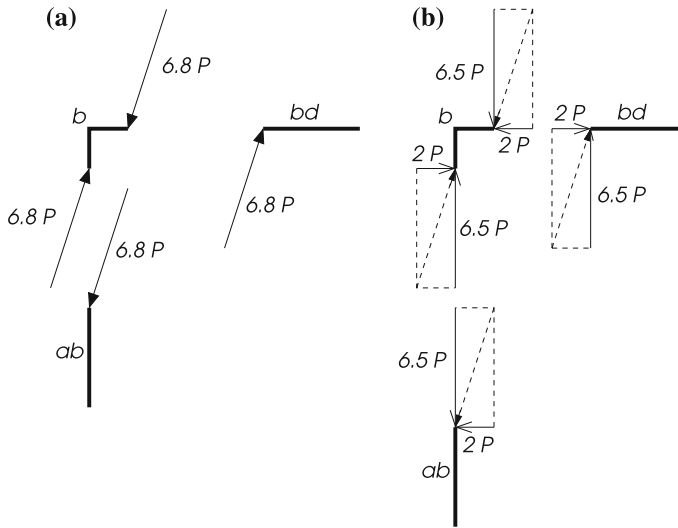


Fig. 2.20 s and n interchange at a corner

along the elements connected to the corner (node) change their names. This is clearly visualized in Fig. 2.20a. The resultant of the internal forces (stresses) acting on both sides of sections b_L and b_R is $\sqrt{2^2 + 6.5^2}P = 6.8P$.

The horizontal component of this force is a *normal* internal force $2P$ in beam bd and a *shear* force $2P$ in column ab (Fig. 2.20b). Similarly, the vertical component of the force is a *shear* force $6.5P$ in beam bd and a *normal* force $6.5P$ in column ab .

2.8.3 Why Tensile Side for Bending Moments?

Beams are primarily bending elements and as such have tension along the bottom fibers and compression along the upper fibers when the bending moment is positive. The opposite is true for segments with negative bending.

When you design beams made of reinforced concrete it is very important to know where the tensile stresses are. Concrete is an artificial composite material made of an aggregate of cement, stone, sand and water. It is a rather easy material to come by and, similar to rock, it behaves very well in compression but not in tension. Mountain ranges have peaks which measure in kilometers, their bases incur very heavy compressive forces (self weight) and still they are perfectly stable, if not for the tectonic forces. Arguably, the heaviest man-made structure, if it can be called a structure, the Great Pyramid of Cheops in Upper Egypt may have lost its erstwhile glitter but its base still holds the punishing pressure of over two million huge blocks of limestone. The same prevails for concrete.

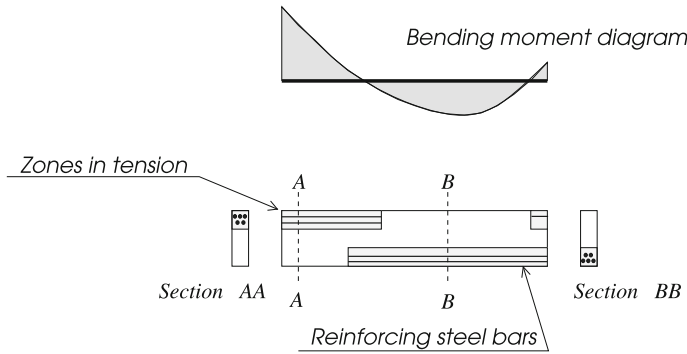


Fig. 2.21 Reinforced concrete example

Like rock, concrete is brittle in tension. Pull on a concrete bar and chances are that it will crack in no time. This is why we use reinforced concrete in our structures. Steel is very effective in tension and steel bars are embedded in the concrete in regions where tension is expected. Consequently, when designing a beam made of reinforced concrete we will position the reinforcing bars near the fibers in tension, that is, *on the side where we draw the bending moment*.

The parabolic bending moment diagram in Fig. 2.21 is typical for beams under distributed uniform forces. We notice that to the left of the point of zero bending the upper fibers are in tension while the lower ones are in compression and to the right of that point the lower fibers have tensile stresses. The diagram is telling us to position the reinforcing bars in the correct zone. Starting from the left, the steel reinforcing bars should hug the upper surface of the beam cross-section and then move to the underside tracking the tensile stresses. (The drawing is of course very schematic.)

It has been argued that in countries where concrete beams are the rule the diagrams are on the tensile side, where concrete is poised to crack, whereas in regions where the beam are usually made of steel, the bending moment is on the compressive side. Steel beams have more to fear from compression, which may cause local buckling and wrinkling of the flanges, than from tensile stresses.

2.8.4 How Are the Applied Forces Annulled?

We should say a word regarding the direct equilibrium of the external forces. Here, we need opposite internal forces applied exactly at the external forces to annul them. (Note, the internal forces will be read off the nsm diagrams in Fig. 2.6.) Consider a thin lamella at the base of column ab (node a in Fig. 2.22). On one side it has two applied forces and an applied couple (white arrows) and on the other side, inside the material, the internal forces (black arrows). This is trivial.

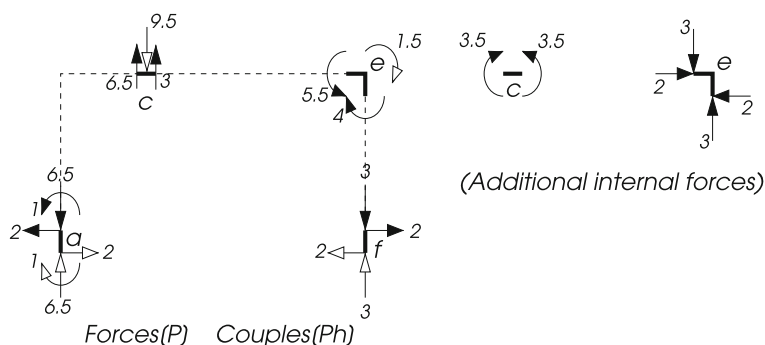


Fig. 2.22 Equilibrating the applied loads in a frame

Section c is a little more interesting. Taking the values off the shear force diagram in Fig. 2.6, we find that the $9.5P$ force is annulled by the sum of the shear forces on both sides of the thin lamella beneath the force.

At corner e , we have a similar situation where the applied $1.5Ph$ couple is annulled by the bending moments 5.5 and $4.0Ph$.

Finally, at f the shear and normal forces annul directly the two applied forces.

2.8.5 Noise on the Way to Equilibrium

At node c and corner e , we can get a glimpse of how structures behave (see the additional drawings in Fig. 2.22. At e , for instance, the applied couple is indeed balanced but by a difference of two couples, each of which is, however, about three times larger than the applied one. We would have preferred smaller bending moments to do the job, but this is the only way the structure is capable of annulling the external forces. In fact, most of the internal forces are unwanted guests. All we want is to annul the applied forces, and the rest, as far as we are concerned, is superfluous. But the frame in Fig. 2.6 cannot do a better job than that.

Finally:

How do structures produce the internal forces? By deforming.

But this is for later.

2.9 Illustrated Examples

2.9.1 Frame with Point Loads

Again consider frame $abcdef$ in Fig. 2.4. We will now attempt to determine the $nsm(x)$ shown in Fig. 2.6 (in a frame we have all three types of internal forces). You will have noticed that the positive axes are bottom-up for columns ab , fe and left-right for beam be . It is subjected to point forces and point couples in self-equilibrium. Segments ab , bc , ce , ef are without loads and cover the entire length of the frame. Consequently, the shear force on every segment is constant and the bending moment is linear. Thus, we need one value per segment for the shear force and two for the linear bending moment (usually chosen close to the extremities). It is by now clear that we compute the internal forces on the $+$ facet of every virtual cut.

In Fig. 2.23, we have computed the internal forces on the $+$ facet at a_R , b_L (just in front of corner b), b_R (right after corner b), c_L (right in front of the point load at c), c_R (right after the point load) and e_L . Note, when passing from e^- to e^+ the normal force 2.0 becomes a shear force, and the shear force 3.0 is now a normal force. The bending moment drops from 5.5 to 4.0 because of the concentrated couple at the corner.

These results are universal. They are true all over the world. The way we are displaying the results, however (see Fig. 2.6), is region-dependent. In this text we use the results in Fig. 2.23 to draw the diagrams in Fig. 2.6 as follows.

- For the *shear diagram* we recall that the shear is drawn in the physical direction of the shear force on the $+$ facet (right facet) of the section. In the first instance, we need to indicate the begin and end directions by means of arrows. The arrows

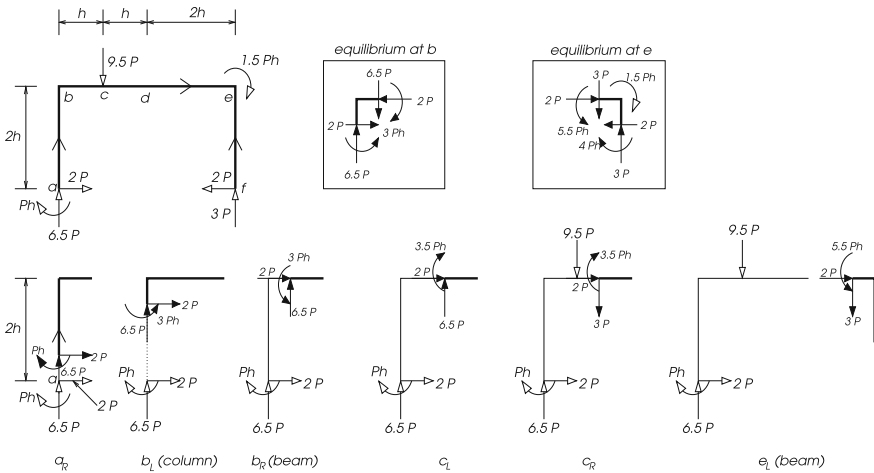


Fig. 2.23 Calculating nms in frame

on the columns were chosen in the up direction and the arrow on the beam is, of course, pointing right. For column ab starting from a , we have a shear force on the $+$ facet of any section pointing inside the frame, consequently the constant shear is drawn on the inside. On segment bc , we have at b_R^+ a shear force $6.5P$ pointing up, consequently the shear diagram is there above the line. On segment ce , on the other hand, at c_R^+ we have a down force $3P$ and the shear diagram is there under the line (down) with magnitude $3P$. Finally, the direction of column fe was chosen from f to e as indicated by the arrow. Starting from the left, that is f , we note that the shear force on a $+$ facet points inwards and thus the constant shear diagram is there above the line (or inside the frame).

- For the *moment diagram* we remember that it is composed of straight lines which follow the tensile side of the beam. For instance, on ab the moment at a (Ph) causes tension in the fibers inside the frame (see sketch ' a_R ') and at b ($3.0Ph$) the fibers on the outside of the frame are tensile (see ' b_L '). Hence the moment diagram on segment ab in Fig. 2.6 is a straight line running from inside the frame at a to outside the frame at b .

From b to c we go from tension above the axis ($3.0Ph$) to tension below the axis ($3.5Ph$). On ce the moment diagram starts with tension inside at c ($3.5Ph$) to tension outside at e ($5.5Ph$).

- Finally, the *normal forces diagram* is drawn below the axis for tensional (that is positive) forces and above the line for compressive forces. Here the stresses are uniform over the section and are either in tension or in compression. 'Above' and 'below' have the same meaning as in the shear diagram. (We define the axis direction and, if it is horizontal, the beginning should be at our left.) The representation of the normal forces in the normal diagram in Fig. 2.6 is self-evident. Incidentally, the normal forces in this example are all compressive.

It is important to keep in mind that of the three diagrams the most important by far is the moment diagram.

2.9.2 Distributed Forces Between Point Loads

The beam in Fig. 2.24 is subjected by downwards uniformly distributed forces over the left half of the span, two clockwise point couples and two forces at the extremities. One can check that the loading is auto-equilibrated, that is, the applied forces and couples are statically equivalent to zero but, they induce internal forces which we plan to determine.

We already know that along ac the shear is linear and the bending moment parabolic, and along ce the shear is constant (we recall that shear diagrams are not affected by point couples), and the bending moment has a constant slope (because the shear is constant), but is discontinuous due to the presence of point couples, as can be seen in the shear and moment diagrams of Fig. 2.24. As always the beam and matching sm diagrams are drawn one beneath the other.

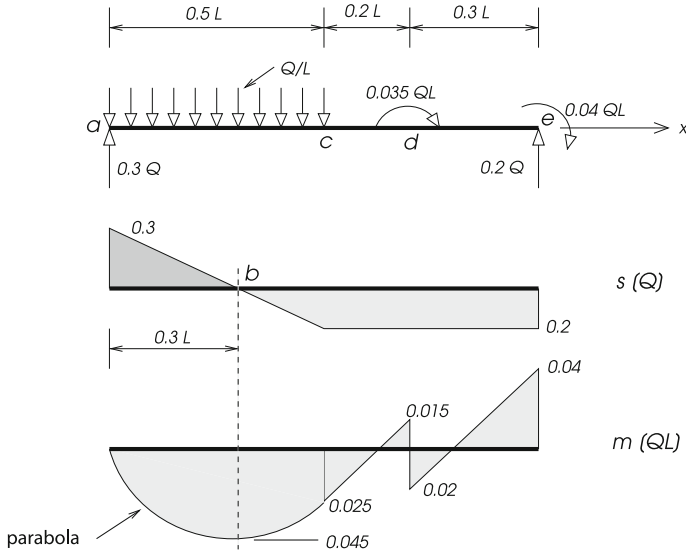


Fig. 2.24 sm diagrams with distributed forces

What is left to compute are the extreme values of the diagrams. In the present case, we will follow a different route for computing the internal forces. We divide the beam into nodes where point loads are applied (a , d and e) and segments with constant distributed forces. This requires another node c where the magnitude of the distributed forces changes from Q/L to zero. The segments are thus a_Rc , cd_L and d_Re_L (see the exploded view in Fig. 2.25).

We now determine the internal forces at the $+$ facets by using the equilibrium conditions, starting with node aa_R^- then cascading to $a_R^+c^-$ and so on until e_L^+ . The individual sketches are shown in Fig. 2.25 and are self-explanatory. For every sketch, the known forces are denoted by white arrows and the ones that are computed are shown with black arrows. (There are at most two black arrows in every diagram, and we have only two equilibrium equations to compute them.)

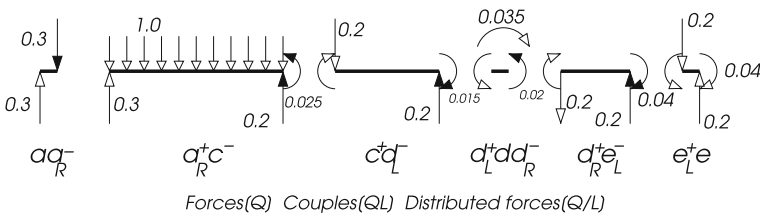


Fig. 2.25 Segments end loads

At extremity a , the external force $0.3Q$ is balanced by the shear force. The latter is calculated, hence the arrow is black. For segment $a_R^+c^-$ the same shear force as at a^+ is now known and is white, while the shear and moment at c^- are computed (black), etc. (At node d we have ignored the shear forces in the diagram for reasons of clarity.)

Note, when we reach e all the forces are known (white). There is nothing you can do regarding the equilibrium of e only hope that it works out well. This is a verification (necessary but not sufficient) that everything has been computed accurately. If e is in equilibrium, there is a good chance that everything is in order. If e is not in equilibrium, something is wrong. We can now draw the sm distributions.

We can draw the diagrams for every segment separately and glue them together or do it in one stretch from left to right. The results are in Fig. 2.24. We draw the m diagram below the s diagram to follow the correlation between loads, shear and moment. Moving information from one diagram to the other is both a verification and an aid in establishing the distributions.

For the bending moment, we have determined the end-values on the interval ac , and since the shear force passes through zero in the interval we also evaluate the moment there. Indeed, $dm/dx = -s$, and when $s = 0$, the derivative of the moment is zero, in other words, the moment is at an extremum.

We can easily determine the coordinate of x_b where the shear is zero. The right-angled triangles built on segments ab and bc are similar and consequently $x_b/0.3 = (0.5L - x_b)/0.2$ or $x_b = 0.3L$. This is also the point of extremum (largest) bending moment.

There are several ways to compute the moment at b . One of them is by integrating $sdx = -dm$, or, $\int_a^b sdx = -\int_a^b dm = m_a - m_b$. In our case $m_a = 0$ and therefore $m_b = -\int_a^b sdx$, or, m_b is equal to the area of the triangle built on segment ab , that is, $m_b = -(0.3L \times -0.3Q)/2 = 0.045QL$ (the shaded area in the s -diagram of Fig. 2.24). As for the sign, you can verify that s above the line is algebraically negative, so is the integral. The moment is thus positive and therefore drawn below the line.

We must admit that juggling with the signs and coordinate axes is an error-prone activity. It is good practice to use the area method to determine the unsigned value of the bending moment and then rely on your engineering judgment as to where the moment should be drawn.

Note that the slope of the bending moment at c depends on s_c only ($dm/dx = -s$). The slope is thus the same on the parabola at c^- and on the straight line at c^+ . In other words, the straight line which is the bending moment along cd is the same as the slope of the parabola at c .

This should be enough to draw an adequate representation of the bending moment diagram.

2.9.3 Inverse Analysis of a Frame

It is often much easier to reconstruct the applied external loads from the $nsm(x)$ distributions than the other way around. The following example is a simple illustration. It is also a good exercise for checking whether you have read the nsm diagrams correctly.

Given the nsm distributions for the frame in Fig. 2.26 what would the applied loads be? (Notice the arrows in the sm distributions.) In a first instance, the ns on segments ab , bc and cd are piecewise constant; consequently there are no distributed applied forces on these segments. Next we look for discontinuities at the limits of the segments from which we deduce the presence of point loads.

At extremities a and d the normal and shear values are the applied loads there. The bending moments are zero, so there are no applied couples at the extremities (Fig. 2.27b). At node b , we note discontinuities in both the n and s distributions. The sketch in Fig. 2.27a is one way to compute the external forces causing the discontinuities. The black arrows are internal forces read from the nsm diagrams at b . The white arrows are computed external forces which maintain equilibrium, written conveniently in the axial, lateral and rotational directions. (We also show in Fig. 2.27a that corner c is perfectly balanced by the internal forces so there are no external loads there.)

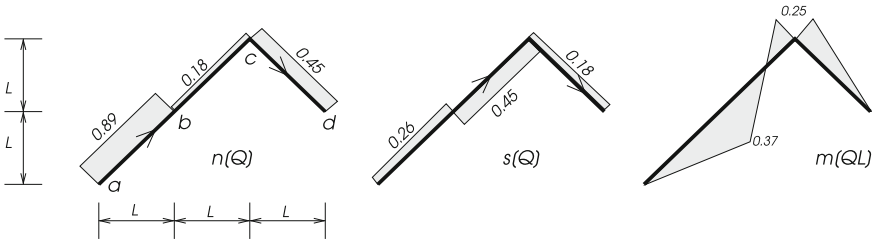


Fig. 2.26 Inverse analysis: the nsm distribution

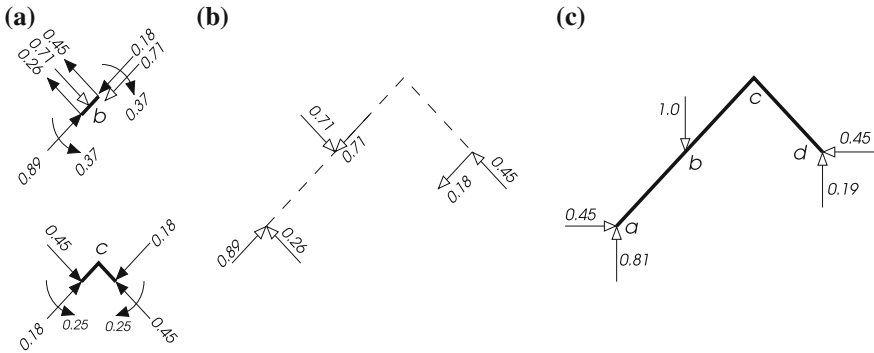


Fig. 2.27 Inverse analysis: calculating the loads

The results are shown in Fig. 2.27b. When we study the displacement method, such components of the forces will be called ‘local’ whereas the ones in Fig. 2.27c will be ‘global’.

Finally, you were right to note that the s diagram is superfluous. It can be obtained from the m diagram ($s = -dm/dx$).

2.9.4 Kinked Bar

Consider the kinked bar⁴ in Fig. 2.28a subjected to co-linear forces F at a and c . The element can be considered as having been straight of length $2L$ and having received a kink in its middle.

When the bar was straight ($h = 0$) the internal forces reduced to a compressive axial forces $n(x) = -F$ (no shear, no bending moments). What happened when the element was bent by a kink at b ? First and foremost we have now internal bending moments, which is not a good omen.

Indeed, in Fig. 2.28b we have depicted half the structure, ab as a free body. (Since the structure and the loading are symmetric, so is the solution. The forces on the other half are the mirror image.) At a we have F . Somewhere else there must be an equal and opposite force, if not, the body under consideration would accelerate to the right. Since the half-element is devoid of any additional external forces, we must assume that there is an equal and opposite internal force F at the right end b of the (now) free body. This force is inside the structure, and there we can surmise anything plausible. So far translational equilibrium is satisfied. But there is now an anti-clockwise couple Fh (the two forces F at a distance h). We need a clockwise couple Fh to maintain rotational balance and this can only be the bending moment inside the element at b . The free body is now in equilibrium.

To draw the diagrams of the internal forces, it is convenient to decompose the end forces in axial and lateral components as in Fig. 2.28c. We can now draw the internal forces diagrams. Since there are no distributed forces, and the element is straight, the normal and shear distributions are constant. We will note that $n(x) = -F \cos \alpha$ and $s(x) = F \sin \alpha$ (α is the slope of the bar) are both drawn below the axis. The bending moment is linear, joining the value at a (zero) to Fh at b (tension above). This covers the left part of the diagrams. We note, as expected that the normal internal forces n are reduced with α and the shear s and bending moments m are increased.

The right part is their mirror images, but for the shear diagram. We are reminded that the shear diagram of a symmetric problem⁵ is anti-symmetric! The actual shear forces are of course symmetric (Fig. 2.29a), but because the x -coordinate runs from left to right on both sides of the axis of symmetry, the shear diagram is anti-symmetric (Fig. 2.29b).

⁴We usually reserve the term ‘bar’ for a straight element under axial loads and ‘beam’ for an element in bending. The kinked bar is something which was a bar and acquired characteristics of a beam.

⁵A symmetric problem is a symmetric structure with symmetric loads.

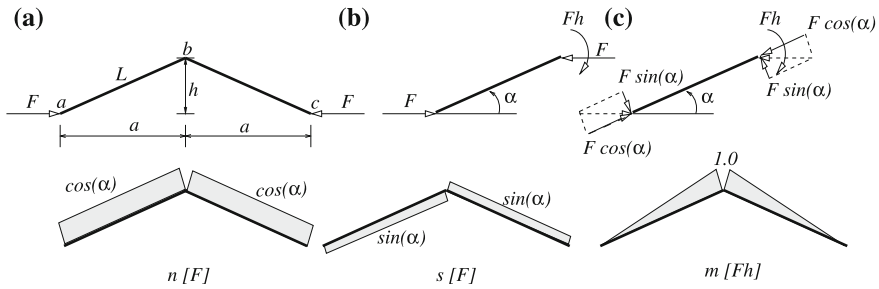


Fig. 2.28 nsm of kinked bar under compressive forces

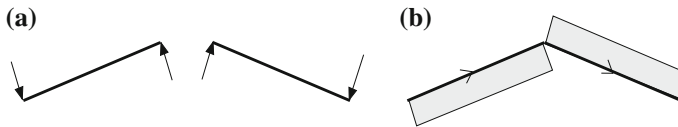


Fig. 2.29 Symmetric shear forces; anti-symmetric $s(x)$. **a** Shear forces (symm.). **b** Shear diagram (anti-symm.)

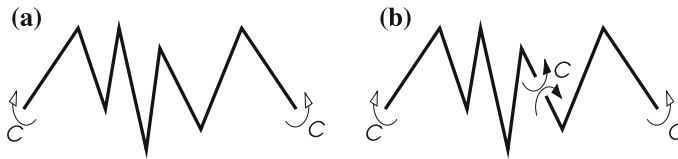


Fig. 2.30 Zigzag frame

2.9.5 Broken Line with End-Couples

We shall conclude on a light note. The zigzag (rigid) frame in Fig. 2.30a is subjected to equal and opposite couples C at its extremities. What are the nsm ?

Take a section anywhere and you will note that all you need to enforce equilibrium is a positive bending moment of magnitude C (Fig. 2.30b). Therefore, $n = s = 0$ and $m = C$ (tension below the line) everywhere.

2.10 Summing Up

We have concluded a somewhat lengthy description of how to compute the internal forces in line segments, and more so of how to depict them in $nsm(x)$ diagrams. You will have noticed that it is all an exercise in equilibrium.

Typical cases were shown and some examples solved. The *nsm* diagrams permeate the field of structures, and will appear repeatedly throughout this and other texts on structural analysis and design.

Learning is a repeating process, done in ever widening circles. The reader will have many opportunities to either return to this chapter or to improve her/his skills by studying and drawing other *nsm(x)* diagrams which will appear further on.

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